

Holographic Wilsonian RG flow and Sliding Membrane Paradigm

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We study the relations between two different approaches to the holographic Renormalization Group (RG) flow at the dual gravity level: One is the radial evolution of the classical equation of motion and the other is the flow equation given by the holographic Wilsonian RG coming from the cut off independence. Apparently, the two flows look different. We give general proofs that the two flows are actually equivalent. The role of the momentum continuity (MC) is essential. We show that MC together with cutoff independence gives the evolution equation of the boundary values. Equivalence of conductivity flows in two paradigm has been shown as an explicit example. We also get the connecting formula of Green functions and AC conductivity at arbitrary slice in terms of its value at horizon for various geometry backgrounds.

I. INTRODUCTION

From the early days of gauge/gravity dualities [1–3], the role of radius as the energy scale was emphasized explicitly and recent work on fermion dynamics in charged AdS₄ [4–6] also showed that the radial region trapping the fermions can play the role of fermi sea, demonstrating that the radial scale really encodes the energy scale of the boundary dynamics. Therefore interpreting the radial evolution of the classical solution as the renormalization group flow has been tempting and it was originally encoded in the jargon “scale radius duality” [7] or “UV/IR correspondence” [9] and it was sharpened in ref. [8], where the flow was written in terms of classical equation of motion and the Callan-Symanzik equation was written in the context of AdS/CFT.

In the presence of the black hole, one can recast the problem in the following way. There has been two different computations for transport coefficients; one at the horizon [15] and the other at the boundary [16]. To see the relations between the two holographic screens, one may try to introduce a cutoff surface at a finite radial position r_c and try to view two holographic screens as special cases of running cutoff surface. In this framework, one can calculate the flow of physical quantities like conductivity [14] or diffusion constant [13] explicitly. In this way, the RG flow can be quantitatively formulated in finite temperature context, which is a nontrivial procedure in field theory set up.

More recently, authors in the papers [11, 12, 17], considered the Wilsonian renormalization group flow [10] by dividing the radial direction into high (UV) and low (IR) energy scales and integrating out the UV part. As a result, the procedure results in flow equations. The

question is whether the ‘traditional’ holographic RG flow coming from the classical equation of motion [7, 8, 13, 14] is the same as the more recently formulated one in [11, 12, 17]. Partial answer to this question was already given in [11, 12] by showing that solution to classical equation of motion can be used to construct the quadratic order of effective action.

In this paper, we point out that holographic Wilsonian RG equation in gravity limit is equivalent to flow equation coming from the equation of motion. We first show this in general context and also explicitly demonstrate it for the case of the conductivity. We will also derive an effective action with momentum correction up to second order. With a low frequency effective action at hand, we obtain the relation of the Green functions at two different cut off surfaces, which can be viewed as the integrated version of the RG flow of the Green function. Finally, we will show that this relation precisely gives the double trace flow of Green function if we integrate out UV geometry and it also precisely gives the transport coefficients flow which have been obtained in sliding membrane paradigm if we integrate out IR geometry. We also give the solution of AC conductivity from a constant value at horizon to another constant at the boundary. From the point of view in deconstructing holography, by integrating out UV geometry, one obtained UV action of the effective coupling and a cutoff dependent dispersion relation for Goldstone boson can be obtained.

II. SLIDING MEMBRANE AND HOLOGRAPHIC WILSONIAN RG: A REVIEW

In this section, we review some of the known results for RG flow to set up our questions. We use the notation of [14, 17]. The $d+1$ dimensional metric is $ds^2 = -g_{tt}dt^2 + g_{rr}dr^2 + g_{ij}dx^i dx^j$. We assume the above metric has a horizon at $r = r_H$ and UV boundary at $r = \infty$. Except that g_{tt} has a first order zero and g_{rr} has a first order pole at the horizon, all other g_{ij} are finite. $\{\mu, \nu\}$ run over d dimensional space time. We will use both r and z where r is used for the radial coordinate and z is used for the inverse of it.

Sliding Membrane: Cutoff-dependent transport coefficients were defined on a sliding membrane in [14]. Such diffusion constant was worked out in [13]. In-falling horizon condition is shown to be equivalent to the regular horizon condition in Eddington-Finkelstein coordinates. Let us take a bulk U(1) gauge field with standard Maxwell action for example. All the quantities on the cutoff membrane can be derived from an outer surface action $S_b = \int_{r_c} d^d x j^\mu A_\mu$, where $j^\mu \equiv \Pi^\mu = -\sqrt{-g}F^{r\mu}$. Since one can visualize that the theory is defined on the cutoff surface which can be anywhere, it can be called as a sliding membrane. That is, the source and the conjugated momentum can be defined for any r_c and k_μ . Assume the momentum is along the i direction, then the longitudinal r_c dependent conductivity can be defined by $\sigma(r_c, k_\mu) = \frac{j^i(r_c, k_\mu)}{F_{it}(r_c, k_\mu)}$. From now on we delete subindex c from r_c if it is not confusing. The equation of motion requests that $\sigma(r_c, k_\mu)$ satisfies the flow equation [14]:

$$\partial_r \sigma = i\omega \sqrt{\frac{g_{rr}}{g_{tt}}} \left[\frac{\sigma^2}{\Sigma(r)} \left(1 - \frac{k^2 g^{ii}}{\omega^2 g^{tt}} \right) - \Sigma(r) \right], \quad \Sigma(r) = \frac{\sqrt{-g} g^{ii}}{\sqrt{g_{rr} g_{tt}}}. \quad (1)$$

The initial condition for this differential equation is not arbitrary since regularity condition at the horizon requests that $\sigma(r_H) = \Sigma(r_H) = \frac{\sqrt{-gg^{ii}}}{\sqrt{g_{rr}g_{tt}}} \Big|_{r=r_H}$. This result is consistent with the definition $\sigma(r_c, k_\mu) = \frac{j^i(r_c, k_\mu)}{F_{it}(r_c, k_\mu)}$ applied at the horizon.

Deconstructing Holography: In the work of [13, 14], the dependence of physical quantities on the running cutoff implies the idea of renormalization group in the context of the holography. However, it was not very clear how the integrating out the high frequency degree of freedom in the Wilsonian sense is implemented. For this purpose, D.Nickel and D.T.Son [17] proposed a method to integrating out the UV dynamics. The idea is the deconstruction: replacing the 5 dimensional UV action of $S = S_{\text{IR}}(r < r_c) + S_{\text{UV}}(r_c < r)$ by an effective 4 dimensional action: for the case of Maxwell action, the 4 dimensional action proposed was [17]

$$S_{\text{UV}} = \frac{1}{2} \int d^4x [f_\mu (\partial_\mu \varphi - a_\mu + A_\mu)^2] , \quad (2)$$

where the goldstone field $\varphi = \int_{r_c}^\infty A_r(r, x)$ is coming from the breaking the two $U(1)$ carried by the two boundary fields $a_\mu = A_\mu(r_c)$ and $A_\mu = A_\mu(\infty)$ and f_μ is given by metric factors. a_μ is external source of the theory at r_c , which is proposed to be determined from the current balance condition

$$\frac{\delta S}{\delta a_\mu} = \frac{\delta S_{\text{UV}}}{\delta a_\mu} + \frac{\delta S_{\text{IR}}}{\delta a_\mu} \equiv j + j_{\text{IR}} = 0 . \quad (3)$$

This equation together with the current conservation equation $\partial_0 j^0 + \partial_i j^i = 0$ and the definition of the conductivity $\sigma \equiv \frac{j_{\text{IR}}}{f_{i0}} = \frac{j}{\partial_0 a_i - \partial_i a_0}$ gives the diffusion equation in the low frequency limit $\partial_0 j^0 - \frac{\sigma}{f_t} \partial_i \partial^i j^0 = 0$.

At this moment, we want to pose a problem. The sliding membrane paradigm is to get the radial dependence of physical quantity from the classical equation of motion and interpret the cut off dependence as the RG flow. On the other hand, deconstruction is to obtain the radial dependence of physical quantity by integrating out the UV geometry. Are these two cut off dependence of transport coefficients consistent?

Holographic Wilsonian RG In [11], it was suggested how to identify the bulk and boundary path integrals in Wilsonian approach where the whole boundary path integral is splitted into two parts:

$$Z = \int_{\text{boundary}} DM_{k\delta < 1} DM_{k\delta > 1} e^{-S} = \int_{\text{boundary}} DM_{k\delta < 1} e^{-S(\delta)} := \langle \exp(-s(\delta)) \rangle ,$$

where δ is the cut-off length scale and $S(\delta) = S_0 + s(\delta)$. One can derive RG equation from the δ independence of the whole integral. The object of the Wilsonian RG is to obtain $s(\delta)$. The holographic version of this is supposed to be implemented by splitting the radial scale into two and write the bulk partition function accordingly:

$$Z = \int_{\text{bulk}} D\phi_{z > \epsilon} D\phi_{z = \epsilon} D\phi_{z < \epsilon} e^{-S_{\text{IR}}(z > \epsilon) - S_{\text{UV}}(z < \epsilon)} = \int_{\text{bulk}} D\phi_\epsilon \Psi_{\text{IR}}(\epsilon, \tilde{\phi}) \Psi_{\text{UV}}(\epsilon, \tilde{\phi})$$

with $\tilde{\phi} = \phi(z = \epsilon)$. Notice that we are using inverse coordinate $z = 1/r$ and $\epsilon = 1/r_c$. The question is how to identify the two path integral at the level of split? The idea of [11] is that one should identify the $\Psi_{IR}(\epsilon, \phi_{z=\epsilon})$ as the dual path integral with a UV cut off on a scale δ with a complete set of single trace operators and the effective action of UV dynamics, $S_B = -\log \Psi_{UV}(\epsilon, \phi_{z=\epsilon})$, can be determined from the independency of the cutoff or from the semiclassical approximation. As a consequence, one can identify

$$e^{-s(\delta)} = \int D\tilde{\phi} e^{\int d^d x \tilde{\phi}(x) \mathcal{O}(x)} e^{-S_B}. \quad (4)$$

For later use, we review the U(1) gauge field example in $d + 1$ dimensional fixed background following [12]. The action is given by $S = S_1[z > \epsilon, A_M] + S_B[A_M, \epsilon]$, with $S_1 = -\frac{1}{4} \int_{z>\epsilon} d^{d+1}x \sqrt{-g} F_{MN} F^{MN}$. Here S_B is the boundary action containing all the information of integrating out UV physics. The boundary momentum at $z = \epsilon$ for S_1 should satisfy the condition $\Pi_\mu \equiv -\sqrt{-g} F^{z\mu} = \frac{\delta S_B}{\delta A_\mu}$. The presence of the boundary action S_B is attributed to the multi-trace deformation [11, 12, 19]. See [20–25] for earlier discussions. The flow equation was worked out [12] by requiring $\frac{d}{d\epsilon} S = 0$:

$$\partial_\epsilon S_B[A_\mu, \epsilon] + H_1[A_\mu, \frac{\delta S_B}{\delta A_\mu}] = 0, \quad (5)$$

where H_1 is the hamiltonian of the IR action evaluated at $z = \epsilon$. The quadratic action (2) is a solution satisfying flow equation of S_B and the corresponding effective action in boundary field theory can be obtained by doing Legendre transform of S_B [12]:

$$I_{\text{eff}} = \int \left[-\frac{1}{2} \left(\frac{1}{f_t} (j^0)^2 - \frac{1}{f_i} (j^i)^2 \right) - j^\mu (A_{b,\mu} + \partial_\mu \varphi) \right]. \quad (6)$$

We will see this boundary effective action can be derived by use of the deformed Green function later.

It is interesting to ask whether the presence of the S_B can give any effect to the classical path at all. In order to clarify these questions, we shall first generally demonstrate the equivalence between holographic Wilsonian RG flow in bulk classic level and radial flow of equation of motion. This also can solve our problem mentioned before for the relation between sliding membrane and deconstructing holography due to the same structure between deconstruction and holographic Wilsonian RG.

III. EQUIVALENCE OF HOLOGRAPHIC WILSONIAN RG AND SLIDING MEMBRANE PARADIGM

There are two RG flows. One is the flow given by the classical equation of motion, and the other is the flow from integrating out the geometry. Here we want to prove the equivalence of the two. Start with the full quantum path integral in gravity side and do the classical approximation

$$Z_1[\phi_0] = \int_{\text{bulk}} D\phi e^{-S} \simeq \exp(-S[\phi_c]) \quad (7)$$

Here ϕ_0 is the value of bulk field at the UV boundary and ϕ_H is a boundary condition at the horizon or at IR boundary. The latter can be a regularity condition at the horizon or infalling boundary condition for the probe field case. We emphasize that it is not necessarily a Dirichlet boundary condition at the horizon. ϕ_c is the solution of the classical equation of motion for the given boundary condition. Once two boundary conditions are specified classical field ϕ_c is defined for all z and one can evaluate the physical quantities at any slice. For example we can define the current and electric field and therefore the conductivity thereon. Such quantity will depends on the position of the slice and this is what we mean by the RG flow in sliding membrane. Basically *the flow is given by the bulk classical equation of motion*.

Now we introduce the splitting the path integral into UV and IR as before

$$Z_2 = \int_{\text{bulk}} D\tilde{\phi} \Psi_{\text{IR}}(\epsilon, \tilde{\phi}) \Psi_{\text{UV}}(\epsilon, \tilde{\phi}), \quad (8)$$

and do classical approximation of each part:

$$\Psi_{\text{IR}}(\epsilon, \tilde{\phi}) = \int_{\text{bulk}} D\phi_{z>\epsilon} e^{-S(z>\epsilon)} = e^{-S_{\text{IR}}[\phi_c^{IR}]} \quad (9)$$

$$\Psi_{\text{UV}}(\epsilon, \tilde{\phi}) = \int_{\text{bulk}} D\phi_{z<\epsilon} e^{-S(z<\epsilon)} = e^{-S_{\text{UV}}[\phi_c^{UV}]}, \quad (10)$$

where $S(z > \epsilon) = \int_{\epsilon}^{z_H} dz L[\phi]$ and similarly $S(z < \epsilon) = \int_0^{\epsilon} dz L[\phi]$. Previously we wrote S_{UV} as S_B assuming that it can be expressed explicitly as boundary values only. It is easy to show that when the original action is quadratic in field variables, S_{UV} is actually quadratic form of the two boundary fields. To appreciate the meaning, we should be careful about the boundary condition (BC) of each field ϕ_c^{IR} and ϕ_c^{UV} . The former is the solution of classical equation of motion of the action S between $\epsilon < z < z_H$ with the given boundary conditions ϕ_H and $\tilde{\phi}$ at horizon and at $z = \epsilon$. So it is better to write it as $\phi_c^{IR}[\phi_H, \tilde{\phi}]$. The former is the solution with BC $\tilde{\phi}, \phi_0$ at $z = \epsilon$ and $z = 0$ respectively. So we write it as $\phi_c^{UV}[\tilde{\phi}, \phi_0]$. Notice that so far $\tilde{\phi}$ is completely arbitrary and independent of ϵ . Now we define

$$S_{\epsilon}[\phi_H, \tilde{\phi}, \phi_0] = S_{\text{IR}}[\phi_c^{IR}[\phi_H, \tilde{\phi}]] + S_{\text{UV}}[\phi_c^{UV}[\tilde{\phi}, \phi_0]]. \quad (11)$$

Then we can write Z_2 as a functional integral and take classical approximation again.

$$Z_2 = \int D\tilde{\phi} e^{-S_{\epsilon}[\tilde{\phi}]} = e^{-S_{\epsilon}[\phi_H, \tilde{\phi}^*, \phi_0]} \quad (12)$$

where $\tilde{\phi}^*$ is the solution of the equation

$$\frac{\delta S_{\epsilon}}{\delta \tilde{\phi}} = \frac{\delta S_{\text{UV}}}{\delta \tilde{\phi}} + \frac{\delta S_{\text{IR}}}{\delta \tilde{\phi}} = 0, \quad (13)$$

which is the condition of the minimizing the $S[\tilde{\phi}]$. Since $\Pi = \pm \frac{\delta S}{\delta \tilde{\phi}}$ depending on whether ϵ is upper or lower boundary of the z integral of the action, we have

$$\Pi_{\text{UV}} = \frac{\delta S_{\text{UV}}}{\delta \tilde{\phi}}, \quad \Pi_{\text{IR}} = -\frac{\delta S_{\text{IR}}}{\delta \tilde{\phi}}, \quad (14)$$

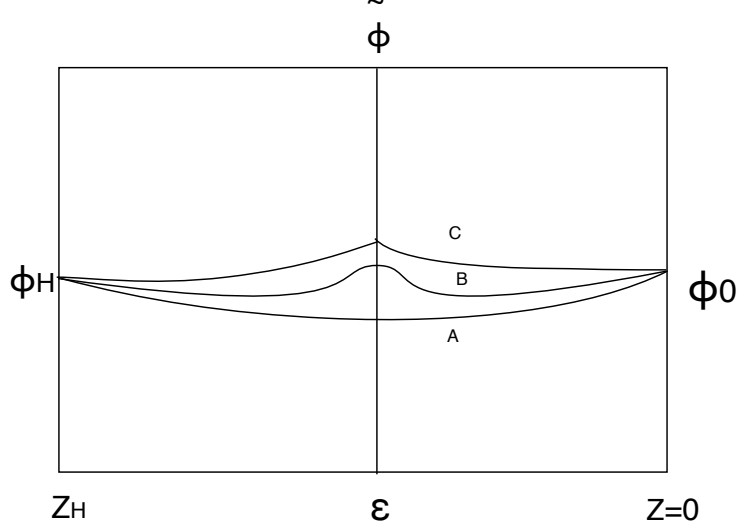


FIG. 1: Flows with (A) and without (C) the momentum continuity. The uniqueness of the smooth solution forbids a solution like B. Therefore classical version of integration over $\tilde{\phi}$ pick up the $\tilde{\phi} = \phi_c(z)|_{\epsilon}$. The uniqueness of A means that classical flow of S_{UV} or S_{IR} in Z_2 along ϵ can only pick up the path A.

which implies that the above eq. (13) is nothing but the continuity of momentum across the $z = \epsilon$.

$$\Pi_{IR} = \Pi_{UV}. \quad (15)$$

What is the solution $\tilde{\phi}^*$? Given the boundary values ϕ_H and ϕ_0 , the bulk field $\phi(z)$ is determined for entire region $z_H > z > 0$ by the classical equation of motion, which will give the value of Z_1 . Notice that ϕ_{IR}^* (which is determined by ϕ_H and $\tilde{\phi}^*$) should be joined with ϕ_{UV}^* (which is determined by ϕ^* and ϕ_0) to give a classical solution ϕ^* . In order for $Z_1 = Z_2$, it is sufficient to have

$$\tilde{\phi}^* = \phi_c(z)|_{z=\epsilon} \text{ so that } \phi^* = \phi_c. \quad (16)$$

Notice that $\tilde{\phi}$ is originally defined as a boundary value at $z = \epsilon$ and it is independent of ϵ . We should also notice that (16) guarantees the ϵ independence of the Z_2 and S_ϵ , i.e., $\frac{dS_\epsilon}{d\epsilon} = 0$, because with the solution (16) S_ϵ becomes the $S[\phi_c]$, the classical value of action of Z_1 which is manifestly independent of ϵ . So far we have shown that classical solutions satisfy the requirement of ϵ independence of split partition Z_2 . This is the reason why the solution of the Wilsonian RG equation can be written in terms of the solution of classical equation of motion.

Conversely, we want to argue that the continuity of the momentum (CM) implies that eq.(16) holds. The CM means that the two solutions ϕ_{IR} and ϕ_{UV} are smoothly joined. Without CM, there is a solution (with a cusp) which connects ϕ_H, ϕ_0 through $\tilde{\phi}$. See the curve C of Fig.1. Let's denote by A the curve describing ϕ_c that smoothly connect the given boundary conditions ϕ_H and ϕ_0 , and suppose the solution of the momentum continuity $\tilde{\phi}^*$ do not agree with $\phi_c(\epsilon)$. Let's denote by B the classical path that connect ϕ_H and ϕ_0 through

$\tilde{\phi}^*$. The momentum (or velocity) continuity means that curve B in Fig.1 is also a smooth classical path with the same boundary condition of the curve A, which is contradiction to the uniqueness of the solution of differential equation. The uniqueness of A means that classical flow of S_{UV} or S_{IR} in Z_2 along ϵ can only pick up the path A. These arguments establish the equivalence of the classical radial flow and the Wilsonian RG flow.

It is useful to see explicitly how the ‘boundary conditions’ $\tilde{\phi}$ at each slice are patched together to agree with the solution of radial equation of motion. Actually it is the role of Wilsonian flow equation that controls how the boundary condition $\tilde{\phi}$ should change as one changes the membrane location.

So, we want to derive the radial evolution for $\tilde{\phi}$ from RG equation. For doing this we relax the S_{IR} to be off-shell by considering the action before doing path integral $D\phi_{IR}$ but after $D\phi_{UV}$. Then, by requesting the ϵ independence at this stage, one can easily establish the flow equation for the S_{UV} [11, 12]

$$\partial_\epsilon S_{UV}[\tilde{\phi}, \phi_0] + H[\tilde{\phi}, \Pi_{\tilde{\phi}}] = 0. \quad (17)$$

$\Pi_{\tilde{\phi}}$ is defined by $\Pi_{\tilde{\phi}} = \frac{\delta S_{UV}[\tilde{\phi}, \phi_0]}{\delta \tilde{\phi}}$ and so equal to Π_{UV} . The functional form of the Hamiltonian H originally was a functional of the bulk field $\phi(z)$ and its momentum conjugate $\Pi(z)$. It becomes functional of $\tilde{\phi}$ by evaluating it at $z = \epsilon$ and it becomes a functional of $\Pi_{\tilde{\phi}}$ using the momentum continuity. Taking the derivative of eq. (17) with respect to $\tilde{\phi}$ we get

$$\partial_\epsilon \Pi_{\tilde{\phi}} = -\frac{\delta H}{\delta \tilde{\phi}}. \quad (18)$$

The first term of this equation depends on $\tilde{\phi}$ which is not ϵ dependent apparently so that the equation can be a second order one.

The origin of the ϵ dependence of the ‘boundary value’ $\tilde{\phi}$ is again coming from the momentum continuity. The point is that while Π_{UV} depends only on the boundary value $\tilde{\phi}$, Π_{IR} is given by the evaluation of the bulk field at $z = \epsilon$: $\Pi_{IR} \equiv \frac{\delta L_{IR}}{\delta \phi}|_\epsilon = -\sqrt{-g}g^{zz}\partial_z\phi(z)|_\epsilon$ for most of the second derivative Lagrangian so that the momentum continuity equation can also be written as

$$\Pi_{\tilde{\phi}} = -\sqrt{-g}g^{zz}\partial_z\phi(z)|_\epsilon = -\sqrt{-g}g^{zz}\partial_\epsilon\tilde{\phi}, \quad (19)$$

giving the ϵ dependence of $\tilde{\phi}$.

Clearly (18) and (19) give the radial evolution for $\tilde{\phi}$. The equations (18) and (19) together with ϕ_0, ϕ_H as the boundary condition of $\tilde{\phi}$ repeat the whole classical solution. This establishes the equivalence of the Wilsonian flow equation and the classical equation of motion. It should be noticed that (17) is usually used to find the form of effective action $S_B = S_{UV}$ but here its functional derivative is used to give the flow equation for the boundary value $\tilde{\phi}$.

The above proof goes through for vector and tensor fields. Therefore we do not repeat the proof here.

Although integrating out degree of freedom in terms of classical geometry, these correspond to the quantum process in the boundary theory. Integrating the geometry gives

deformation of the IR action in a way determined by the classical radial evolution, which can be interpreted as the double trace deformation, whose dominance in turn should be attributed to the large N nature.

A. Example: Flow of Conductivity

In this subsection, we will point out the equivalence of holographic Wilsonian RG equation and the conductivity flow in slicing membrane paradigm. We will derive the membrane flow equation of conductivity (1) from the Wilsonian RG equation (5). For simplicity, we start by assuming S_B is quadratic form of A_μ and consider only the longitudinal mode. In order to keep the same notation with [12], we use the metric

$$ds^2 = -g_{tt}dt^2 + g_{ii}d\vec{x}^2 + g_{zz}dz^2 . \quad (20)$$

S_B was assumed as the form [12] with $\hat{A} = A - \partial\phi$ by

$$S_B[\hat{A}_\mu, \epsilon] = -\frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \sqrt{-\gamma} \left(f_0(k, \epsilon) \hat{A}_0(k) \hat{A}_0(-k) + f_L(k, \epsilon) g^{ii} \hat{A}^L(k) \hat{A}^L(-k) \right) , \quad (21)$$

which can be rewritten in our language as

$$S_B = -\frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \left(-G^{00}(k, \epsilon) \hat{A}_0(k) \hat{A}_0(-k) + G^{ii}(k, \epsilon) \hat{A}_i(k) \hat{A}_i(-k) \right) , \quad (22)$$

where we assume the coefficient of mix term of \hat{A}_0 and \hat{A}_i approximately vanishes ($f_{0i} \sim 0$). From (5) with A replaced by \hat{A} , by comparing coefficients one can obtain the flow equations for G^{00} and G^{ii} [12]:

$$\begin{aligned} \partial_\epsilon G^{00} &= -\frac{(G^{00})^2}{\sqrt{-g}g^{tt}g^{zz}} + \sqrt{-g}g^{tt}g^{ii}k^2 \\ \partial_\epsilon G^{ii} &= -\frac{(G^{ii})^2}{\sqrt{-g}g^{ii}g^{zz}} - \sqrt{-g}g^{tt}g^{ii}\omega^2 . \end{aligned} \quad (23)$$

The definition of conductivity is given by

$$\sigma \equiv \frac{J^i}{E_i} = \frac{J^i}{-\partial_0 \hat{A}_i + \partial_i \hat{A}_0} . \quad (24)$$

With the definition of current $J^\mu \equiv \frac{\delta S_B}{\delta A_\mu}$, from (22) we have

$$-\frac{1}{G^{00}} = \frac{\hat{A}_0}{J^0}, \quad \frac{1}{G^{ii}} = \frac{\hat{A}_i}{J^i} . \quad (25)$$

Using conservation equation $\partial_0 J^0 + \partial_i J^i = 0$, (24) can be presented in momentum space as

$$\frac{1}{\sigma} = \frac{i}{\omega} \left(\frac{\omega^2}{G^{ii}} - \frac{k^2}{G^{00}} \right) . \quad (26)$$

Using (23) and (26), we obtain

$$-\frac{\partial_\epsilon \sigma}{i\omega} = \left(\sigma^2 \left(\frac{1}{\sqrt{-g}g^{ii}g^{zz}} - \frac{k^2}{\omega^2} \frac{1}{\sqrt{-g}g^{tt}g^{zz}} \right) - \sqrt{-g}g^{tt}g^{ii} \right), \quad (27)$$

where we used $g^{tt}g^{ii}k\omega \sim 0$ which comes from the flow equation of approximately vanishing f_{0i} . Under coordinate transformation $z = 1/r$, one has

$$\partial_\epsilon = -r^2 \partial_r, \quad \sqrt{-g}_{(z)} g^{tt} g^{ii} = \sqrt{-g}_{(r)} g^{tt} g^{ii} r^2, \quad \frac{1}{\sqrt{-g}_{(z)} g^{zz} g^{ii}} = \frac{1}{\sqrt{-g}_{(r)} g^{rr} g^{ii}} r^2. \quad (28)$$

we see that (27) precisely gives the flow equation (1) in sliding membrane paradigm.

B. RG Solutions: second order of momentum

In this subsection, we want to find solutions for the RG equations (23). We start by assuming

$$\frac{1}{G^{00}} = \frac{1}{G_{(0)}^{00}} + k^2 \frac{1}{G_{(1)}^{00}} + \dots, \quad \frac{1}{G^{ii}} = \frac{1}{G_{(0)}^{ii}} + \omega^2 \frac{1}{G_{(1)}^{ii}} + \dots \quad (29)$$

and

$$\partial_\epsilon G_{(0)}^{00} = -\frac{(G_{(0)}^{00})^2}{\sqrt{-g}g^{tt}g^{zz}}, \quad \partial_\epsilon G_{(0)}^{ii} = -\frac{(G_{(0)}^{ii})^2}{\sqrt{-g}g^{ii}g^{zz}}. \quad (30)$$

We have the following solutions by imposing simple vanishing boundary conditions for $G_{(1)}^{00}, G_{(1)}^{ii}$

$$\frac{1}{G_{(1)}^{00}(z)} = \int_0^z -\frac{\sqrt{-g}g^{tt}g^{ii}}{G_{(0)}^{00}}, \quad \frac{1}{G_{(1)}^{ii}(z)} = \int_0^z \frac{\sqrt{-g}g^{tt}g^{ii}}{G_{(0)}^{ii}}. \quad (31)$$

We see that k^2 and ω^2 correction are controlled by zero order solutions in the low frequency approximation.

IV. FLOW SOLUTIONS FROM INTEGRATING OUT GEOMETRY

Now we shall discuss the flow solutions for Green function coming from integrating out geometry over arbitrary region.

A. Integrating out $z_0 < z < \epsilon$ and Effective S_B

We first derive an effective action by directly integrating out geometry of $z_0 < z < \epsilon$ part, where $z = z_0$ and $z = \epsilon$ are two cutoff surfaces. We start from the Maxwell dynamics in the following $d+1$ dimensional general background

$$ds^2 = g_{tt}dt^2 + g_{ii}d\vec{x}^2 + g_{zz}dz^2, \quad (32)$$

with the metric components only depending on z . Note that we can relate this metric to that in review section by $z = \frac{1}{r}$ and we have no minus before g_{tt} here. Maxwell equations are written by

$$\partial_z [\sqrt{-g} (\partial^z A^\mu - \partial^\mu A^z)] + \partial_\nu [\sqrt{-g} (\partial^\nu A^\mu - \partial^\mu A^\nu)] = 0. \quad (33)$$

We describe the zero momentum limit case here and leave the momentum corrected results in appendix A. We assume

$$A_\mu(z, k) = A_\mu^{(0)}(z) + k^2 A_\mu^{(1)}(z) + k_\mu k^\nu A_\nu^{(2)}(z) + \dots \quad (34)$$

and A_z is independent on k . In the small momentum $\partial\partial \rightarrow 0$ limit, the first term will dominate and the above equation can be solved by

$$\partial_z \left[A_\mu^{(0)} - \partial_\mu \int_{z_0}^z A_z dz \right] = \frac{C_1^\mu}{\sqrt{-g} g^{zz} g^{\mu\mu}}, \quad (35)$$

where $z = z_0$ is the initial position with $A_{\mu, z_0} = A_\mu^{(0)}(z_0)$ fixed. By defining $\varphi(x^\mu, z) = \int_{z_0}^z A_z dz$, the gauge invariant field $\hat{A}_\mu^{(0)} = A_\mu^{(0)} - \partial_\mu \varphi$ can be solved in terms of the boundary condition of the gauge field at the $z = z_0$:

$$\hat{A}_\mu^{(0)}(z) - A_\mu^{(0)}(z_0) = \int_{z_0}^z \frac{C_1^\mu}{\sqrt{-g} g^{zz} g^{\mu\mu}} dz. \quad (36)$$

Remain task is to determine C_1^μ . At the other boundary $z = \epsilon$, we also need to give a boundary condition. Since we know that we usually have to assign a Dirichlet boundary condition at the infinite boundary as a rule of AdS/CFT correspondence (standard quantization), we choose the same type of boundary condition at ϵ slice. Given $\hat{A}_\mu^{(0)}(z = \epsilon)$, C_1^μ is determined to be

$$C_1^\mu = \frac{1}{\int_{z_0}^\epsilon \frac{dz}{\sqrt{-g} g^{zz} g^{\mu\mu}}} (\hat{A}_\mu^{(0)}(\epsilon) - A_{\mu, z_0}^{(0)}) := f_\mu (\hat{A}_\mu^{(0)}(\epsilon) - A_{\mu, z_0}^{(0)}) \quad (37)$$

with $\frac{1}{f_\mu} = \int_{z_0}^\epsilon \frac{1}{\sqrt{-g} g^{zz} g^{\mu\mu}} dz$ and μ runs over d dimension.

Now, we want to integrate out z in region $z_0 < z < \epsilon$. For the standard quadratic Maxwell action, we obtain the on-shell action as boundary term using the equations of motion:

$$S_{[z_0, \epsilon]}^{\text{on-shell}} = -\frac{1}{2} \int d^d x \sqrt{-g} g^{zz} g^{\mu\mu} \hat{A}_\mu^c \partial_z \hat{A}_\mu^c \Big|_{z_0}^\epsilon = -\frac{1}{2} \int d^d x C_1^\mu \hat{A}_\mu^c \Big|_{z_0}^\epsilon. \quad (38)$$

where we used $\partial_z \hat{A}_\mu^{(0)} = \frac{C_1^\mu}{\sqrt{-g} g^{zz} g^{\mu\mu}}$. Using solution (37), we finally obtain the zero order on shell action

$$S_{[z_0, \epsilon]}^{\text{on-shell}} = -\frac{1}{2} \int d^d x \sum_\mu f_\mu (\hat{A}_\mu(\epsilon) - A_{\mu, z_0}) (\hat{A}_\mu(\epsilon) - A_{\mu, z_0}). \quad (39)$$

This expression was suggested first in [17] as UV effective action without proof. The same result has been obtained for pure AdS case in [6], while the above discussion works for general diagonal backgrounds. The same result was given in [12] without explicit derivation. We discuss in the general diagonal geometry background with zero U(1) background charge, which also can contain the world volume of D brane.

B. Green Function Flow

With an effective action at hand, we want to find the relation between two Green functions defined at the two surfaces. In a background with a horizon at $z = z_H$, one way to define the retarded Green functions at $z = \epsilon$ [14] is to assume that the source and operator relation is the same with that at the $z = 0$ which is discussed in [14, 18]. The definitions of currents are given by

$$J_{z_0}^\mu \equiv \frac{\delta S_{[z_0, \epsilon]}^{\text{on-shell}}}{\delta A_{\mu, z_0}} = J^\mu, \quad J_\epsilon^\mu \equiv \frac{\delta S_{[z_0, \epsilon]}^{\text{on-shell}}}{\delta A_{\mu, \epsilon}} = -J^\mu. \quad (40)$$

For simplicity, we only consider Green function in the diagonal case in linear response

$$G_{z_0}^{\mu\mu} = \frac{J_{z_0}^\mu}{A_{\mu, z_0}}, \quad G_\epsilon^{\mu\mu} = -\frac{J_\epsilon^\mu}{A_{\mu, \epsilon}}. \quad (41)$$

Since the current at left and right boundary of the $z_0 < z < \epsilon$ zone is the same in the magnitude and opposite in direction, such z independence of the current magnitude gives a flow equation of the Green function. The structure of the effective on shell action dictates that Green function should satisfy

$$\frac{1}{G_\epsilon^{\mu\mu}} - \frac{1}{G_{z_0}^{\mu\mu}} = \frac{1}{f_\mu} + \frac{\partial_\mu \varphi}{J^\mu}. \quad (42)$$

It describes the holographic flow of Green function since both z_0 and ϵ can be any places outside the horizon.

1. $z_0 = 0, \epsilon \ll z_H$, Related to Double Trace Flow

When $z_0 = 0$ is fixed, the region we integrated becomes exactly the UV region and only the surface $z = \epsilon$ is left, which should correspond to UV cut-off length scale in Wilson's description of quantum field theory by a certain way. In the saddle point approximation, the whole bulk action becomes

$$S = S_1 + S_{[0, \epsilon]}^{\text{on-shell}}. \quad (43)$$

$S_{[0, \epsilon]}^{\text{on-shell}}$ can be treated as a boundary action on the bulk dynamics of $z > \epsilon$ region. The quadratic boundary term is attributed to the double trace deformation term. The form of Green function with double trace deformation can be generally written as [12]

$$G_\kappa = \frac{1}{G_{\kappa=0}^{-1} + \kappa}, \quad (44)$$

where κ is the double trace coupling and $G_{\kappa=0}$ is the Green function without double trace deformation. In order to relate (44) to our (42), note that when $\kappa = 0$, (44) makes Green function well define at $z = 0$ without double trace deformation. This is the well-known

Green function defined at $z = 0$ in usual AdS/CFT, corresponding to the z_0 boundary Green function in our discussion, which implies (up to renormalization)

$$G_{\kappa=0} = G_{z_0=0}^{\mu\mu} . \quad (45)$$

Along with it, the double trace deformed Green function can be given by (up to renormalization)

$$G_{\kappa} = G_{\epsilon}^{\mu\mu} . \quad (46)$$

Thus the effective action in boundary field theory can be restored from

$$\delta S_{\text{eff}} = -\frac{1}{G_{\kappa}} \delta J \quad J = -\frac{1}{G_{\epsilon}^{\mu\mu}} \delta J_{\epsilon}^{\mu} J_{\epsilon}^{\mu} . \quad (47)$$

where J is field theory operator and A is the corresponding source. Here we establish $J = J^{\mu}$ and $A = A_{\mu,\epsilon}$. Using (47) and (42) we obtain

$$S_{\text{eff}} = \int d^d x \sum_{\mu} \left[-\frac{1}{2} (J^{\mu})^2 \frac{1}{f_{\mu}} - J^{\mu} (A_{\mu,0} + \partial_{\mu} \varphi) \right] . \quad (48)$$

We see that it precisely gives the boundary effective action (6) derived in [12], which provides a consistent check for the above discussions. Note that in order to keep the same direction with the current corresponding the usual boundary value of gauge field, it is natural to define $J^{\mu} = -J_{\epsilon}^{\mu}$ is the dual current operator in field theory corresponding to source $A_{\mu,\epsilon}$. As a closing remark for this subsection, the flow equation given in (42) can be related to the double trace deformation of the holographic Green function with help of (45) and (46).

2. $z_0 = 0$, $\epsilon < z_H$, Related To Deconstruction

In this subsection we again fixed $z_0 = 0$, while keeping ϵ not far away from the horizon. The reason is that we want to find long distance version of holographic liquid. We have proved in the low frequency limit, on shell action over $0 < z < \epsilon$ is (39) with $z_0 = 0$, which precisely gives the construction of S_{UV} in deconstructing holography [17].

We start from the full action ($A_{\mu,b} = A_{\mu,0}$)

$$S = S_1 + S_{0 < z < \epsilon}^{\text{on-shell}} . \quad (49)$$

Apparently, S_1 is the holographic part, which has a finite UV boundary $z = \epsilon$. To simplify the discussion, we assume $A_{\mu,b}$ vanishes at this moment. As usual real time calculation in AdS/CFT, on shell S_1 equals to a boundary term at ϵ (with the gauge $A_z = 0$ and horizon in-falling condition)

$$+ \frac{1}{2} \int d^d x \sqrt{-g} g^{zz} A_{\mu} g^{\mu\mu} \partial_z A_{\mu} \Big|_{\epsilon} . \quad (50)$$

With the help of $G^{\mu\mu} = \sqrt{-g} \partial A^\mu / A_\mu$ for the holographic part, we rewrite the action as

$$S_{\text{on-shell}} = +\frac{1}{4} \int_\epsilon d^d x G^{\mu\mu} A_\mu^2 - \frac{1}{2} \int_\epsilon d^d x \sum_\mu f_\mu (A_\mu - \partial_\mu \varphi)^2. \quad (51)$$

This action can be considered as the semi-holographic construction for Maxwell fluctuations, an analogy with the semi-holographic Fermi liquid construction in [6]. From this action, we can solve the dispersion relation for φ , the only excitation in the low frequency limit, once we input correlation information in the holographic part. A novel example has been checked for holographic zero sound in [17] in the limit $\epsilon \sim \infty$ at zero temperature. In principle, we can find the ϵ dependent dispersion relation, which may have more interesting applications.

To relate the Green function flow in this letter to the work [17] by D.Nickel and D.T.Son explicitly, note that the conductivity defined on the ϵ slice in (54) can be equivalent to the IR conductivity $\sigma \equiv \frac{j_{\text{IR}}}{f_{i0}}$ in [17] through a so called momentum balance condition. (Note that in our discussion, in order to get the flow of Green function and conductivity, we do not need such a balance condition.) After dropping $E_{i,z0}$, a diffusion constant can be read from (55)

$$D = \sigma_\epsilon / f_0. \quad (52)$$

The only condition is

$$\partial_0 J^i / f_i \ll \partial_i J^0 / f_0. \quad (53)$$

One observation in [12] is that, even we assume momentum always balance from the two sides of the cutoff surface ϵ , the cutoff surface can not be taken too closed to the horizon, since in that case the condition (53) is broken due to divergent $\frac{1}{f_i}$. We shall study this in the following part.

C. Conductivity Flow

The flow of Green function depends on φ , the massless Goldstone mode suggested in [17]. In the following, we will see the similar flow equation for conductivity which depends only on the background geometry. We start from the standard definition of conductivity on two slices

$$\sigma_{z0} = \frac{J_{z0}^i}{E_{i,z0}}, \quad \sigma_\epsilon = -\frac{J_\epsilon^i}{E_{i,\epsilon}}, \quad (54)$$

with $E_i = -\partial_0 A_i + \partial_i A_0$. Using the identity $\partial_0 \partial_i \varphi = \partial_i \partial_0 \varphi$ we have

$$\frac{1}{\sigma_\epsilon} - \frac{1}{\sigma_{z0}} = \frac{-f_i^{-1} \partial_0 J^i + f_0^{-1} \partial_i J^0}{J^i}. \quad (55)$$

With the conservation equation for J^μ , one can easily rewrite the above equation (55) in momentum space

$$\frac{1}{\sigma_\epsilon} - \frac{1}{\sigma_{z0}} = -\frac{ik^2}{\omega} \frac{1}{f_0} - \frac{i\omega}{f_i} \quad (\partial_i = ik, \partial_0 = -i\omega). \quad (56)$$

In the diffusion region with small $\omega \sim k^2$, (56) becomes

$$\frac{1}{\sigma_\epsilon} - \frac{1}{\sigma_{z_0}} = -\frac{ik^2}{\omega} \frac{1}{f_0}, \quad (57)$$

while in the region $k \rightarrow 0$ first, it becomes

$$\frac{1}{\sigma_\epsilon} - \frac{1}{\sigma_{z_0}} = \frac{-i\omega}{f_i}, \quad (58)$$

which is AC conductivity flow. Apparently, (56) gives the flow for conductivity depending on small k and ω .

1. $\epsilon \rightarrow z_H$, Related To Sliding Membrane Paradigm In Diffusion Region

In this subsection, we will point out that flow of conductivity (56) is equivalent to the sliding membrane flow of conductivity in the diffusion region. Within a diffusion scaling in the low frequency limit and the initial condition at the horizon, the r dependent conductivity can be solved by [14]

$$\frac{1}{\sigma(r)} = \frac{1}{\sigma_H} + i\frac{k^2}{\omega} \int_{r_H}^r dr \frac{1}{\sqrt{-g g_{rr} g_{tt}}}, \quad (59)$$

Apparently, we see that (56) precisely give (59) in the low frequency diffusion region $\omega \sim k^2$, where we should set

$$z_0 = \frac{1}{r}, \quad \epsilon \rightarrow \frac{1}{r_H}, \quad (60)$$

and note that the integral in (59) is invariant under the coordinate transformation $z = 1/r$. In order to clarify how we need to take the above limit, see (56) again and one can find that the only condition for correct

$$\frac{1}{\sigma_{z_0=\frac{1}{r}}} - \frac{1}{\sigma_{\epsilon \rightarrow z_H}} = \frac{ik^2}{\omega} \frac{1}{f_0} \quad (61)$$

is that

$$\frac{\omega}{f_i} \ll 1 \quad (62)$$

holds in the limit $\epsilon \rightarrow z_H$, since in the diffusion region $\omega \sim k^2 \ll 1$, $\frac{k^2}{\omega}$ is order one and $\frac{1}{f_i}$ is finite. It means that low ω should go to zero faster than the horizon limit. Using the horizon transport as an initial condition, one can obtain the running diffusion constants on the sliding surface [13].

As a closing word at this moment, when we set $\epsilon \rightarrow \frac{1}{r_H}$ and z_0 finite, the z_0 slicing surface is equivalent to the sliding membrane defined at r in [14].

2. Flow of AC conductivity

We now turn to the flow of ω dependent conductivity with $k = 0$. For (58), we see that this formula is not regular at the horizon because $\frac{1}{f_i}$ has divergence, thus it is not a complete RG formula for AC conductivity. We will see that the complete RG formula should come from (1). After setting $k = 0$, we have

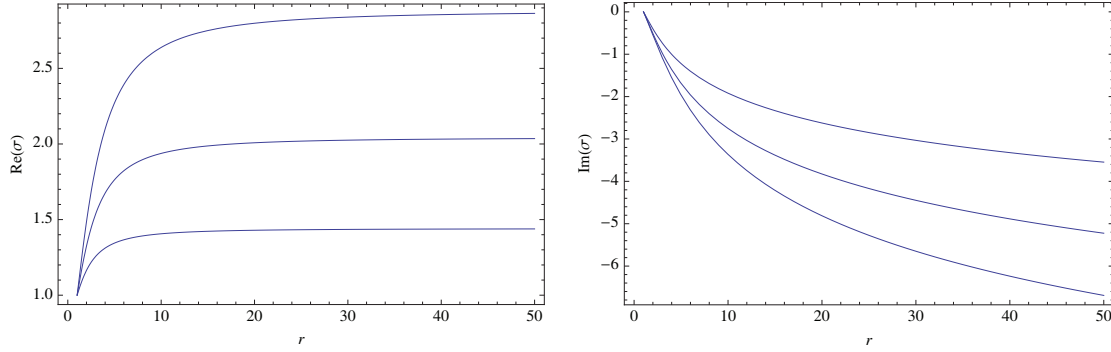


FIG. 2: r flow of AC conductivity, with $d = 4$ AdS-black hole background and $\omega = 2, 1.5, 1$, from up to down in the left Figure and inversely in the right one. For $d > 4$ AdS-black hole, the behavior of solutions are similar.

$$\frac{\partial_r \sigma}{i\omega} = \frac{\sigma^2}{\sqrt{-g}g^{rr}g^{ii}} - \sqrt{-g}g^{tt}g^{ii}. \quad (63)$$

We see that the above equation gives an exact flow for AC conductivity with the initial horizon value, which can be given by requesting equation regular at the horizon. We plot the r flow function of AC conductivity for asymptotical AdS₅ black hole in Figure 2. For $d = 3$, r flow is trivial, and there is no ω dependence for boundary AC conductivity. For other cases, we show the result in the Figure 3,4,5. For the Lifshitz black hole solutions for general z and d , we refer to [26].

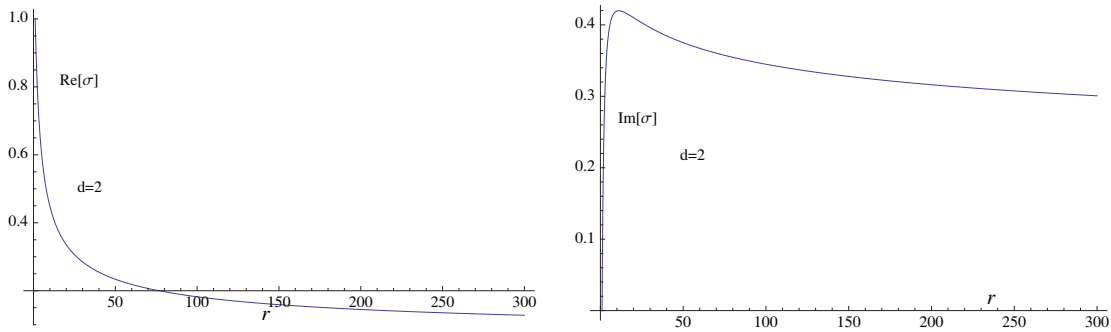


FIG. 3: r flow of AC conductivity, with $d = 2$ AdS-black hole background, $d = 1$ is similar.

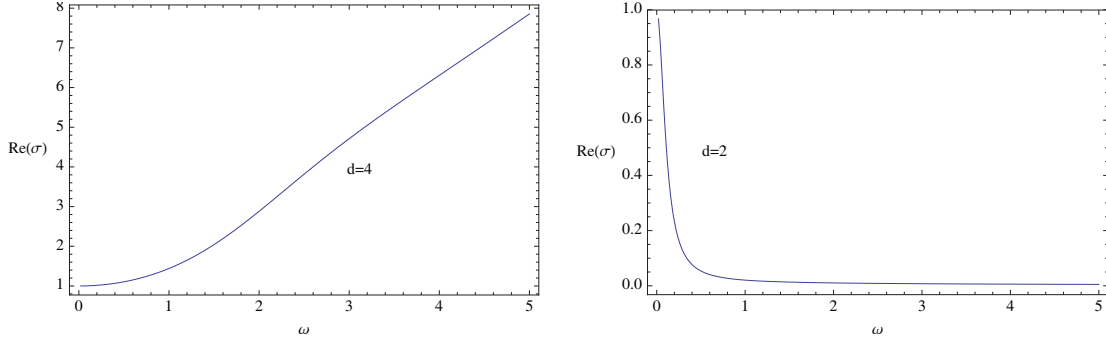


FIG. 4: ω dependence of boundary AC conductivity, with $d = 2, 4$ AdS-black hole background. Behaviors of $d > 4$ case are similar with $d = 4$ and $d = 1$ is similar to $d = 2$.

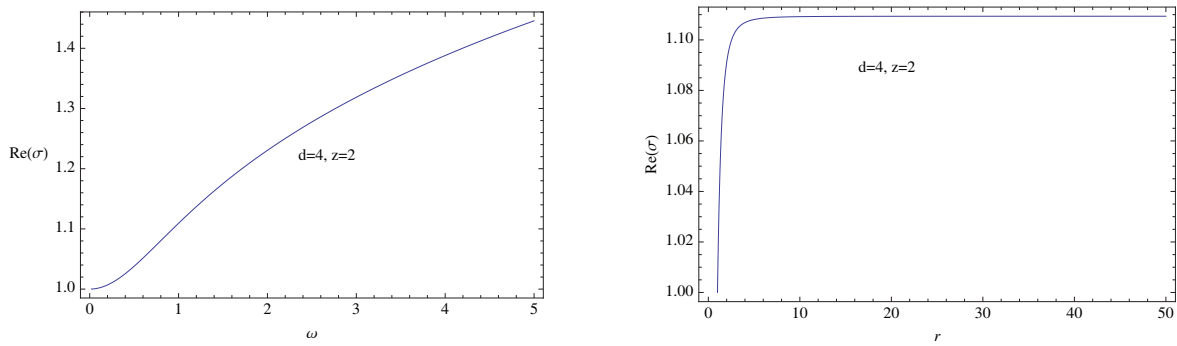


FIG. 5: ω dependence of boundary AC conductivity and r flow of AC conductivity, with $d = 4$ and $z = 2$ Lifshitz-black hole background [26]. Behaviors of $d > 4$ case are similar with $d = 4$.

V. CONCLUSION AND DISCUSSION

To conclude, we would like clarify the relations among deconstructing holography, holographic Wilsonian RG flow and sliding membrane paradigm as follows: Holographic Wilsonian RG equation in bulk classic level is equivalent to radial evolution of equation of motion. The whole transport flow in sliding membrane paradigm can be derived from holographic Wilsonian RG equation. The lesson from the Wilsonian RG is that integrating the geometry gives deformation of the IR action controlled by the classical bulk evolution, which can be interpreted as the double trace deformation.

The method of integrating out geometry over arbitrary region $z_0 < z < \epsilon$ can help to unify deconstructing holography, holographic Wilsonian RG flow and sliding membrane paradigm. S_{UV} part in deconstructing holography can be derived by integrating out UV geometry in low frequency limit. The running dispersion relation of Goldstone boson can be obtained once we know IR holographic correlation precisely. The effective interacting action with finite momentum corrections up to second order has been obtained. In the zero frequency limit, for the UV part $0 < z < \epsilon$, current J^μ is a constant and the flow of deformed Green function

can be derived from the quadratic effective action, which is consistent with the double trace formula of holographic Green function. When we set $\epsilon \rightarrow \frac{1}{r_H}$ and z_0 finite, the z_0 sliding surface is equivalent to the sliding membrane. In the diffusion region, conductivity flow from integrating out geometry method is equivalent to that in sliding membrane paradigm.

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Appendix A: Effective action with finite momentum correction

In order to consider the finite momentum correction, we need to solve (33) without dropping the term with second order derivative. Since the full Maxwell equation is hard to solve in the curved space time we consider the second order derivative only as correction. For convenience, we work in momentum space and take the momentum to be along the i direction. Then Maxwell equations separate into two groups: longitudinal channel and transverse channel. Let us first consider the longitudinal only, then the equation of motion (33) involves

$$\partial_z[\sqrt{-g}g^{zz}g^{00}F_{z0}] + \sqrt{-g}g^{ii}g^{00}\partial_i F_{i0} = 0 , \quad (\text{A1})$$

$$\partial_z[\sqrt{-g}g^{zz}g^{ii}F_{zi}] + \sqrt{-g}g^{ii}g^{00}\partial_0 F_{0i} = 0 . \quad (\text{A2})$$

We expand the A_0 and A_i on small momentum as

$$\begin{aligned} A_0 &= A_0^{(0)}(z) + k_i^2 A_0^{(1)}(z) + k_i \omega A_0^{(2)}(z) + \dots \\ A_i &= A_i^{(0)}(z) + \omega^2 A_i^{(1)}(z) + k_i \omega A_i^{(2)}(z) + \dots \end{aligned} \quad (\text{A3})$$

The term linear in k and ω is deleted in the above expansion due to the structure of equations of motions. Substituting (A3) into equations of motion and using the zero-th order solution (35), one can obtain the following equations for $A_0^{(1)}$, $A_0^{(2)}$, $A_i^{(1)}$, $A_i^{(2)}$ by dropping higher momentum corrections (more than cubic) and assuming k and ω are independent parameters

$$\partial_z[\sqrt{-g}g^{zz}g^{00}\partial_z A_0^{(1)}] - \sqrt{-g}g^{ii}g^{00}A_0^{(0)} = 0 \quad (\text{A4})$$

$$\begin{aligned} \partial_z[\sqrt{-g}g^{zz}g^{00}\partial_z A_0^{(2)}] - \sqrt{-g}g^{ii}g^{00}A_i^{(0)} &= 0 \\ \partial_z[\sqrt{-g}g^{zz}g^{ii}\partial_z A_i^{(1)}] - \sqrt{-g}g^{ii}g^{00}A_i^{(0)} &= 0 \\ \partial_z[\sqrt{-g}g^{zz}g^{ii}\partial_z A_i^{(2)}] - \sqrt{-g}g^{ii}g^{00}A_0^{(0)} &= 0. \end{aligned} \quad (\text{A5})$$

Notice that we have worked out the solutions for $A_0^{(0)}$ and $A_i^{(0)}$, one can easily write the solutions for $A_0^{(1)}$, $A_0^{(2)}$, $A_i^{(1)}$, $A_i^{(2)}$ by solving the above equations. From (36) we have

$$A_0^{(0)}(z) = A_0^{(0)}(z_0) + \int_{z_0}^z \frac{C_1^0}{\sqrt{-g}g^{zz}g^{00}}dz + \partial_0\varphi, \quad (\text{A6})$$

By defining the function

$$F_0(z) = \sqrt{-g}g^{ii}g^{00} \left[A_0^{(0)}(z_0) + \int_{z_0}^z \frac{C_1^0}{\sqrt{-g}g^{zz}g^{00}}dz \right], \quad (\text{A7})$$

one can solve (A4) by

$$\sqrt{-g}g^{zz}g^{00}(\partial_z A_0^{(1)}(z)) = \int_{z_0}^z dz F_0(z) + C_2^{00}, \quad (\text{A8})$$

where C_2^{00} can be determined properly. The final solution for $A_0^{(1)}$ can be given by

$$A_0^{(1)}(z) = \int_{z_0}^z \frac{\int_{z_0}^z F_0(z)dz + C_2^{00}}{\sqrt{-g}g^{zz}g^{00}} + A_0^{(1)}(z_0). \quad (\text{A9})$$

$A_0^{(2)}$, $A_i^{(1)}$, $A_i^{(2)}$ can be determined by the same way

$$A_0^{(2)}(z) = \int_{z_0}^z \frac{\int_{z_0}^z F_i(z)dz + C_2^{i0}}{\sqrt{-g}g^{zz}g^{00}} + A_0^{(2)}(z_0) \quad (\text{A10})$$

$$A_i^{(1)}(z) = \int_{z_0}^z \frac{\int_{z_0}^z F_i(z)dz + C_2^{ii}}{\sqrt{-g}g^{zz}g^{ii}} + A_i^{(1)}(z_0) \quad (\text{A11})$$

$$A_i^{(2)}(z) = \int_{z_0}^z \frac{\int_{z_0}^z F_0(z)dz + C_2^{0i}}{\sqrt{-g}g^{zz}g^{ii}} + A_i^{(2)}(z_0), \quad (\text{A12})$$

where the definitions for $F_i(z)$ is given by

$$F_i(z) = \sqrt{-g}g^{ii}g^{00} \left[A_i^{(0)}(z_0) + \int_{z_0}^z \frac{C_1^i}{\sqrt{-g}g^{zz}g^{ii}}dz \right] + \partial_i\varphi. \quad (\text{A13})$$

Just as we determine C_1^μ in (37), we want to determine the C_2^{00} , C_2^{i0} , C_2^{ii} , C_2^{0i} by the boundary values of gauge field and factors involving integrating out geometry. We use (A9) to determine C_2^{00} , which is obtained as

$$C_2^{00} = \frac{1}{\int_{z_0}^\epsilon \frac{dz}{\sqrt{-g}g^{zz}g^{00}}} \left(A_0^{(1)}(\epsilon) - A_{0,z_0}^{(1)} - \int_{z_0}^\epsilon \frac{\int_{z_0}^z F_0(z)dz}{\sqrt{-g}g^{zz}g^{00}} \right). \quad (\text{A14})$$

In the same way, C_2^{i0} , C_2^{ii} , C_2^{0i} can be obtained as follows

$$C_2^{i0} = \frac{1}{\int_{z_0}^\epsilon \frac{dz}{\sqrt{-g}g^{zz}g^{00}}} \left(A_0^{(2)}(\epsilon) - A_{0,z_0}^{(2)} - \int_{z_0}^\epsilon \frac{\int_{z_0}^z F_i(z)dz}{\sqrt{-g}g^{zz}g^{00}} \right) \quad (\text{A15})$$

$$C_2^{ii} = \frac{1}{\int_{z_0}^\epsilon \frac{dz}{\sqrt{-g}g^{zz}g^{ii}}} \left(A_i^{(1)}(\epsilon) - A_{i,z_0}^{(1)} - \int_{z_0}^\epsilon \frac{\int_{z_0}^z F_i(z)dz}{\sqrt{-g}g^{zz}g^{ii}} \right) \quad (\text{A16})$$

$$C_2^{0i} = \frac{1}{\int_{z_0}^\epsilon \frac{dz}{\sqrt{-g}g^{zz}g^{ii}}} \left(A_i^{(2)}(\epsilon) - A_{i,z_0}^{(2)} - \int_{z_0}^\epsilon \frac{\int_{z_0}^z F_0(z)dz}{\sqrt{-g}g^{zz}g^{ii}} \right). \quad (\text{A17})$$

Now we shall write down the full effective action containing the finite momentum corrections. Note that the first equal sign in (38) holds even at high momentum corrections, one can substitute the solutions $A_0^{(1)}$, $A_0^{(2)}$, $A_i^{(1)}$, $A_i^{(2)}$ into (A3) and evaluating (38). We write the effective action by dropping the higher order momentum corrections as follow

$$\begin{aligned}
S^{\text{onshell}} = S^{\text{on-shell}}_{[z_0, \epsilon]} &+ C_1^0 \left[k_i^2 (A_0^{(1)}(\epsilon) - A_{0,z_0}^{(1)}) + k_i \omega (A_0^{(2)}(\epsilon) - A_{0,z_0}^{(2)}) \right] \\
&+ k_i^2 \left[\int_{z_0}^{\epsilon} F_0(z) dz \times \hat{A}_0^{(0)}(\epsilon) + C_2^{00} (\hat{A}_0^{(0)}(\epsilon) - A_{0,z_0}^{(0)}) \right] \\
&+ k_i \omega \left[\int_{z_0}^{\epsilon} F_i(z) dz \times \hat{A}_0^{(0)}(\epsilon) + C_2^{i0} (\hat{A}_0^{(0)}(\epsilon) - A_{0,z_0}^{(0)}) \right] \\
&+ C_1^{ii} \left[\omega^2 (A_i^{(1)}(\epsilon) - A_{i,z_0}^{(1)}) + k_i \omega (A_i^{(2)}(\epsilon) - A_{i,z_0}^{(2)}) \right] \\
&+ \omega^2 \left[\int_{z_0}^{\epsilon} F_i(z) dz \times \hat{A}_i^{(0)}(\epsilon) + C_2^{ii} (\hat{A}_i^{(0)}(\epsilon) - A_{i,z_0}^{(0)}) \right] \\
&+ k_i \omega \left[\int_{z_0}^{\epsilon} F_0(z) dz \times \hat{A}_i^{(0)}(\epsilon) + C_2^{0i} (\hat{A}_i^{(0)}(\epsilon) - A_{i,z_0}^{(0)}) \right]. \tag{A18}
\end{aligned}$$

This action describes the effective coupling between $A_\mu(\epsilon)$, A_{μ,z_0} and φ including momentum correction up to second order. Flow solutions for transport coefficients from this effective action should be consistent with solution solved from the classical equations of motion under low frequency approximation.

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